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REV. HUMPHREY LLOYD, D. D., President, in the Chair.

Edward Barnes, Esq., and Henry Freke, Esq., were elected Members of the Academy.

The Rev. Charles Graves read the following note on the development of a function in factorials of the variable upon which it depends.

The process of integration for factorials being simpler than that for powers, in the inverse calculus of finite differences, we sometimes have occasion to resolve a proposed function of x into a series of the form

$$A_0 + A_1 x + A_2 x(x-1) + A_3 x(x-1)(x-2) + \&c.;$$

and we may readily determine the coefficients A_0, A_1, A_2, A_3, &c., by making x successively equal to 0, 1, 2, 3, &c. In this way Sir John Herschel, in his *Collection of Examples of the Applications of the Calculus of finite Differences*, has solved the more general problem of developing a function $F(x)$ in a series of factorial terms of the form

$$A_0 + A_1(x-f_1) + A_2(x-f_1)(x-f_2) + \&c.$$

$F(x)$ being any function whatever of x, and f_1, f_2, &c., parti-

cular values of any other function $f(x)$, corresponding to the values 1, 2, &c., of x. The two methods of developing $\mathrm{F}(x)$ in a series of factorials, which are here noticed, seem to have advantages over the method of indeterminate coefficients, in being more simple and direct, and in manifesting more clearly the law which the coefficients A_0, A_1, A_2, A_3, &c., follow. They furnish, at the same time, interesting examples of the use of separating symbols of operations from their operands; and it is for this latter reason, rather than on account of any novelty in the results arrived at, that they are now submitted to the notice of Members of the Academy.

I. Employing u* to denote the operation which changes $\mathrm{F}(x)$ into $\mathrm{F}(x+1)$ we are entitled to write

$$\mathrm{F}(x+n) = u^n \,\mathrm{F}(x) \text{ and } \mathrm{F}(n) = u^n \,\mathrm{F}(o).$$

But u is known to be equivalent to $1 + \Delta$; we may therefore write

$$\mathrm{F}(n) = (1 + \Delta)^n \,\mathrm{F}(o);$$

or, with the right-hand member of the equation developed,

$$\mathrm{F}(n) = \mathrm{F}(o) + \frac{\Delta \mathrm{F}(o)}{1} n + \frac{\Delta^2 \mathrm{F}(o)}{1.2} n(n-1) + \&c. \quad (1)$$

A particular case of this theorem is commonly given in treatises on the calculus of finite differences, viz.:

$$x^n = \frac{\Delta o^n}{1} x + \frac{\Delta^2 o^n}{1.2} x(x-1) + \&c.$$

And indeed the theorem itself may be derived from the fundamental expression for u_{x+n} by making $x = 0$.

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* Arbogast, in his *Calcul des Derivations*, has appropriated the letter E to this use, as being the initial of the word *Etat;* and in so doing he has been followed by recent writers. But against this usage it may be objected that the symbol E is now devoted to a different office in the theory of elliptic functions. And, on the other hand, there seems to be a peculiar fitness in denoting by u that operation which changes u_x into u_{x+1}.

II. If we take the differential coefficient of x^n, and multiply it by x, the result will be $x^n n$; that is to say,

$$x^n n = x\frac{d}{dx}x^n;$$

and as a consequence of this equation, we shall likewise have,

$$x^n \mathrm{F}(n) = \mathrm{F}\left(x\frac{d}{dx}\right)x^n. \qquad (2)$$

In the right-hand member of this equation, let us put $1 + x - 1$ in place of x; and then expand by the binomial theorem; the result will be

$$x^n\mathrm{F}(n) = \mathrm{F}\left(x\frac{d}{dx}\right)x^0 + \frac{\mathrm{F}\left(x\frac{d}{dx}\right)(x-1)}{1}n + \frac{\mathrm{F}\left(x\frac{d}{dx}\right)(x-1)^2}{1.2}n(n-1)$$
$$+ \&c. \qquad (3)$$

The coefficient of $n\,(n-1)\ldots(n-m+1)$ in this development will be

$$\frac{1}{1.2..m}\left\{x^m\mathrm{F}(m) - mx^{m-1}\mathrm{F}(m-1) + \frac{m(m-1)}{1.2}x^{m-2}\mathrm{F}(m-2) - \&c.\right\}$$

and, if we now suppose $x = 1$, we shall have the development of $\mathrm{F}\,(n)$ in the desired form; the coefficient of the factorial $n\,(n-1)\ldots\ldots(n-m+1)$ being

$$\frac{1}{1.2\ldots m}\left\{\mathrm{F}\,(m) - m\,\mathrm{F}\,(m-1) + \frac{m\,(m-1)}{1.2}\,\mathrm{F}\,(m-2) - \&c.\right\}$$

Comparing the two expressions (1) and (3) we find, as we ought to do,

$$\Delta^m\,\mathrm{F}\,(o) = \mathrm{F}\,(m) - m\,\mathrm{F}\,(m-1) + \frac{m\,(m-1)}{1.2}\,\mathrm{F}\,(m-2) - \&c.,$$

a formula which might be obtained directly by making $x = 0$ in the fundamental equation of the calculus of finite differences,

$$\Delta^m u_x = u_{x+m} - mu_{x+m-1} + \frac{m\,(m-1)}{1.2}u_{x+m-2} - \&c.$$

By the aid of the symbol $\left(x\frac{d}{dx}\right)$ we may obtain another interesting development. In virtue of the equation (2) we have

$$e^{hx\frac{d}{dx}}\,x^n = e^{hn}\,x^n = (e^h\,x)^n.$$

It is plain, then, that the symbol

$$e^{hx\frac{d}{dx}}$$

operates on any function of x by changing x into $e^h\,x$; that is to say,

$$\mathrm{F}\,(e^h\,x) = e^{hx\frac{d}{dx}}\,\mathrm{F}\,(x);$$

whence, developing the right hand member, we get

$$\mathrm{F}\,(e^h\,x) = \mathrm{F}\,(x) + \frac{\left(x\frac{d}{dx}\right)\mathrm{F}\,(x)}{1}\,h + \frac{\left(x\frac{d}{dx}\right)^2\,\mathrm{F}\,(x)}{1.2}\,h^2 + \&\mathrm{c}. \quad (4)$$

As Taylor's theorem gives the altered state of $\mathrm{F}\,(x)$, after x has received an increment h, so the theorem just announced exhibits the new value of $\mathrm{F}\,(x)$ after x has been multiplied by a number whose logarithm is h; the series in both cases being arranged according to ascending powers of h.

In executing the operations indicated in the development (4) it must be remembered that $\left(x\frac{d}{dx}\right)^2$ is not equivalent to $x^2\,\frac{d^2}{dx^2}$ but to $x\,\frac{d}{dx}\;x\,\frac{d}{dx}$; and so on for the other powers of the symbol. Neglecting to make this distinction we should get the development of $\mathrm{F}\,(x+xh)$ instead of $\mathrm{F}\,(e^h x)$. The actual relation between the symbols $x^n\,\frac{d^n}{dx^n}$ and $x\,\frac{d}{dx}$ is obtained immediately from the equation (2) which gives us

$$x^n\frac{d^n}{dx^n} = \left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-1\right)\ldots\ldots\left(x\frac{d}{dx}-n+1\right).$$

Sir John Herschel has given the following theorem, which enables us to develope $\text{F}\,(e^h)$ in a series of ascending powers of h when such a development is possible:—

$$\text{F}\,(e^h) = \text{F}\,(1) + \frac{h}{1}\,\text{F}\,(1+\Delta)\,o + \frac{h^2}{1.2}\,\text{F}\,(1+\Delta)\,o^2 + \&c.$$

Comparing this with the one given above, we obtain the following theorem:

$$\left(x\frac{d}{dx}\right)^n \text{F}\,(x) = \text{F}\,[x\,(1+\Delta)]\,o^n,$$

by the help of which we arrive at a still more general one,

$$f\left(x\frac{d}{dx}\right) \text{F}\,(x) = \text{F}\,[x\,(1+\Delta)]\,f(o).$$

Sir William R. Hamilton wished to be allowed to remind the Academy that he had communicated to them, in 1831, another extension of Herschel's Theorem, which was published in the seventeenth volume of the Transactions (page 236), namely, the following:

$$\nabla' f \psi\,(o') = f(1+\Delta)\ \nabla'\,(\psi(o'))^o;$$

where the accents in the first member might have been omitted, and where ∇' denoted any combination of differencings and differentiatings, performed with respect to o', and generally any operation with respect to that accented zero, of which the symbol might indifferently follow or precede $f(1+\Delta)$, as a symbolic factor. By making $\psi\,(o') = \varepsilon^{o'}$, and $\nabla' = \text{D}'^x$, where $\text{D}' = \dfrac{d}{do'}$, the theorem of Herschel is obtained. A much less general formula was cited as "Hamilton's theorem," in the last Number of the Cambridge and Dublin Mathematical Journal, namely, the following:

$$f(x) = f(1+\Delta)\,x^0;$$

which had, however, been also given in the same short paper of 1831.